Identifiability of load models

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Abstract: There have been many models developed to represent electric loads. However, the identifiability problem of load models has not been studied, that is whether the parameters of a model can be identified from a specified input–output experiment. A study is made of the identifiability of electric load models, and the basic concepts and definitions are introduced. Structural identifiability and input identifiability are studied and numerical identifiability is discussed.

1 Introduction

Since about 1941 there has been a continual trickle of technical papers reporting on load characteristics as determined through electrical tests and on the effects of load representation on system dynamic performance. During the past thirty years, there has been an expended interest in load modelling [1–4]. Three kinds of load models have been developed.

(i) Composite induction motor (IM) models [5–11]. Induction motors constitute a large proportion of power system loads. The induction motor models have been long used in analysis and computation.

(ii) Input–output (IO) models [12–18]. This kind of load model was developed in the last decade and is receiving increasing attention.

(iii) Artificial neural network (ANN) models [19, 20], which have recently been proposed.

In the past, considerable attention has been paid to deriving the models of composite loads and their parameter estimation. However, the study on identifiability of load models has not been observed. It is the authors' opinion that identifiability should receive much more attention in developing load models. The main reason is that it is of great practical importance. Those involved in estimating load parameters from measurements would, of course, like to know whether they stand any chance of succeeding. Whenever the parameters are not uniquely identifiable, the very suggestion of an attempt to estimate them is questionable. Many researchers who planned to give it only a passing thought may find themselves trapped into a long-term project.

This paper gives a beginning to work in this area. At first, some basic concepts and definitions are introduced. Then the identifiability problem of load modelling is discussed according to model structure, input type and numerical algorithm.

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2 Concepts and definitions

2.1 Basic concepts

Before turning to formal definitions, an example is presented to illustrate some of the subtleties of the identifiability concept [21]. Consider a simple first-order linear model:

$$\dot{x}(t) = -p_1 x(t) + p_2 u(t)$$
$$x(0) = 0$$
$$y(t) = p_3 x(t) \quad (1)$$

The model has three unknown parameters: p_1, p_2, p_3. For any known input $u(t)$, the explicit solution of eqn. 1 is:

$$y(t) = p_2 p_3 \int_0^t e^{-p_1(t-\tau)} u(\tau) \mathrm{d}\tau \quad (2)$$

If the input is an impulse $u(t) = \delta(t)$, then

$$y(t) = p_2 p_3 e^{-p_1 t} \quad (3)$$

It is well known that a semilogarithmic plot of the data, represented as $y(t)$ for this model, yields the coefficient $a \equiv p_2 p_3$ and exponent $\lambda \equiv p_1$ of this model. Thus only p_1 and the product $p_2 p_3$ can be determined and not p_2 or p_3 separately, i.e. the model is unidentifiable This is also clear from eqn. 2 for any known $u(t)$. If p_2 or p_3 were known, or if a uniquely functional relationship between p_2 and p_3 were known, all parameters could be uniquely determined from $y(t)$, and we could say the model (or model parameters) is (are) uniquely (globally) identifiable.

2.2 Complete model

Let

$X = [x_1\ x_2\ ...\ x_{nx}]^T$ denote the state vector

$U = [u_1\ u_2\ ...\ u_{nu}]^T$ the input

$Y = [y_1\ y_2\ ...\ y_{ny}]^T$ the output (measurement)

$Z = [z_1\ z_2\ ...\ z_{nz}]^T$ the unknown parameter vector.

The observation interval is $t_0 \leq t \leq T$ and we allow the initial state $X_0 = X(t_0, Z)$ to also depend on Z. The (nonlinear) vector function F characterises the known input–state and state–state coupling, and G the known output–state and output–input coupling, each parametrised by Z. Finally, H denotes the vector-valued set of all v additional and independent algebric equality or inequality constraints relating X, U and Z, or any combination of these, known *a priori*. In these terms, the constrained structure, the basic system-experiment model, is given by:

$$\begin{aligned} &\dot{X}(t,Z) = F[X(t,Z),U(t),t;Z] \\ &Y(t,Z) = G[X(t,Z),U(t);Z] \\ &X_0 = X(t_0,Z) \\ &H[X(t,Z),U(t),Z] \geq 0 \\ &t_0 \leq t \leq T \end{aligned} \tag{4}$$

2.3 Identifiability definitions

The single parameter z_i of the model eqn. 4 is:

(*a*) uniquely (globally) identifiable, if there exists a unique solution for z_i from these relationships;

(*b*) locally identifiable, if there exists a countable number (≥ 1) of distict solutions for z_i from eqn. 4 and nonuniquely identifiable if this number > 1;

(*c*) zero unidentifiable, if there exists no solution for z_i from eqn. 4;

(d) ∞ unidentifiable, if there exist an infinite number of solutions for z_i from eqn. 4;

(*e*) interval identifiable if it is ∞ unidentifiable and there exist finite upper and lower bounds z_i^{min} and z_i^{max} from the constrained structure in eqn. 4. The parameter interval is denoted $\Delta z_i = z_i^{max} - z_i^{min}$;

(*f*) quasi-identifiable (*a posteriori*) if it is interval identifiable and Δz_i is small enough to yield a satisfactory 'unique' (point) estimate of z_i for the application at hand;

(*g*) structural unidentifiable if z_i is unidentifiable due to the model structure;

(*h*) input unidentifiable if z_i is unidentifiable due to the form or shape of input signal;

(*i*) numerical unidentifiable if the numerical algorithm used is unable to find the unique solution for z_i from eqn. 4.

The difference between zero unidentifiable and ∞ unidentifiable should be carefully distiguished. Also noting that identifiability results depend on the form or shape of the input functions, the topological structure of the model, as well as the numerical algorithm used.

3 Structural identification

3.1 Example 1: speed dynamic induction motor model

If both stator electrical transients and rotor electrical transients are neglected; furthermore, the magnetising reactance X_m is supposed to be very large, or let $X_m \to \infty$; an induction motor load can be represented by the well known classical speed dynamic model

$$M\dot{s} = T_m - \frac{V^2/X'}{s/s_{cr} + s_{cr}/s} \tag{5}$$

The observable output equations are:

$$\begin{aligned} P &= \frac{V^2}{X'} \frac{1}{s/s_{cr} + s_{cr}/s} \\ Q &= \frac{V^2}{X'} \frac{s/s_{cr}}{s/s_{cr} + s_{cr}/s} \end{aligned} \tag{6}$$

where

M denotes the inertial time constant

s motor slip

s_{cr} critical slip

T_m mechanical torque

X' transient reactance

V voltage imposed on the motor

P, Q active and reactive power respectively.

To investigate the identifiability of this model, a new state variable is defined as

$$x = s/s_{cr} \tag{7}$$

then the state eqn. 5 becomes

$$(Ms_{cr})\dot{x} = T_m - \frac{V^2}{X'}\frac{x}{x^2+1} \tag{8}$$

the output eqn. 6 becomes

$$\begin{aligned} P &= \frac{V^2}{X'}\frac{x}{x^2+1} \\ Q &= \frac{V^2}{X'}\frac{x^2}{x^2+1} \end{aligned} \tag{9}$$

It is clear from eqns. 8 and 9 that only the product M s_{cr} can be determined and not M or s_{cr} separately, i.e. M and s_{cr} are ∞ unidentifiable. It shoud be emphasised that ∞ unidentifiable does not mean there exists no solution but means there exist an infinite number of solutions. Of course, if M or s_{cr} were known any other way, all parameters could be uniquely determined from $\{V(t), P(t), Q(t)\}$.

3.2 Example 2: linear model

Under small disturbance, electric loads can be well described by a linear dynamic model [15]. Let us suppose that the real model is a first-order model represented in the form of a transfer function:

$$y(s) = \frac{a_1}{s-\lambda_1}u(s) \tag{10}$$

If a second-order model is used:

$$y(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2}u(s) \tag{11}$$

To keep the same input–output behaviour, the transfer functions should be the same:

$$\frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} = \frac{a_1}{s-\lambda_1} \tag{12}$$

Hence, the parameters must satisfy:

$$\begin{aligned} \beta_1 &= a_1 \\ \beta_2 - \beta_1\lambda_1 &= a_1\alpha_1 \\ -\lambda_1\beta_2 &= a_1\alpha_2 \end{aligned} \tag{13}$$

Obviously, there exist an infinite number of solutions $\{\beta_2, \alpha_1, \alpha_2\}$. This means that the parameters β_2, α_1, α_2 are ∞ unidentifiable, or the second-order model is ∞ unidentifiable. On the other hand, if the real model is the second-order model and the first-order model is applied to describe it. It is clear from eqn. 13 that there is usually no solution $\{a_1, \lambda_1\}$ for given $\{\beta_2, \alpha_1, \alpha_2\}$, i.e. the first-order model is zero unidentifiable. The above results can be generalised as:

a linear dynamic model is ∞ unidentifiable if its order is higher than actual order, zero unidentifiable if its order is lower than actual order.

It should be pointed out that not only the order but also the form of the model affects the identifiability of model parameters. For example, if α_1 and β_2 in eqn. 13 are set to be zero, then there exists no solution or the model is zero unidentifiable.

4 Input identifiability

4.1 Example 3: composite dynamic–static models

In 1989 [16, 17], a 'composite dynamic–static model (CDSM)' was proposed by the author. This model was further developed in [18]. If the frequency dependency and phase angle dependency are neglected, the model can be written as:

$$\begin{aligned} P &= P_s + P_v \\ T_p \dot{P}_v + P_v &= T_p \dot{g}_v \end{aligned} \tag{14}$$

where T_p is the time constant; $P_s(V)$, $g_v(V)$ are static and transient functions:

$$\begin{aligned} P_s &= P_0 (V/V_0)^{\alpha_s} \\ g_v(V) &= P_0[b_p(V/V_0) + c_p(V/V_0)^2] \end{aligned} \tag{15}$$

It can be proven that if:

$$\begin{aligned} g_v(V) &= K_p(V)/T_p - P_s(V) \\ K_p &= c_p V^2 \end{aligned} \tag{16}$$

the model becomes Hill's model [12]. If:

$$\begin{aligned} g_v(V) &= P_{tl}(V) - P_s(V) \\ P_{tl} &= P_0 (V/V_0)^{\alpha_t} \end{aligned} \tag{17}$$

the model will be the same as Karlsson's model [13], where α_t is the transient exponent. In response to a step voltage variation from V_0 to V_s at $t = 0$, the solution of eqn. 14 consists of two components. One is the quasisteady-state component $P_s(t)$:

$$P_s(t) = P_0 + [P_s(V_s) - P_0]1(t) \tag{18}$$

Another is the voltage-dependent transient component $P_v(t)$:

$$\begin{aligned} P_v(t) &= C_1 e^{-t/T_p} 1(t) \\ C_1 &= g_v(V_s) - g_v(V_0) \end{aligned} \tag{19}$$

In the step test, the quantities P_0, $P(\infty) = P_s(V_s)$ and $P(t)$ are measurable, and hence $P_v(t)$, from which C_1 and T_p can be obtained. Obviously, the static exponent α_s can be derived from $P_s(V_s)$ and P_0. The transient parameter c_p of eqn. 16 or the transient exponent α_t of eqn. 17 can be determined from C_1. Therefore, Hill's model and Karlsson's model are uniquely identifiable with a step test. On the other hand, b_p and c_p (two parameters) cannot be uniquely determined from one value: C_1. Consequently, CDSM (14), (15) is ∞ unidentifiable with step test.

However, we will prove that CDSM is identifiable with ramp voltage variation. The voltage variation is expressed as:

$$V = \begin{cases} V_0 & \text{if } t < 0 \\ V_0 - kt & \text{if } 0 \le t \le t_1 \\ V_s & \text{if } t_1 \le t \le T \end{cases} \tag{20}$$

It is proven in Appendix 9.1 that the solution of CDSM is:

$$P_v(t) = \begin{cases} 0 & \text{if } t < 0 \\ d_1(e^{-t/T_p} - 1) + d_2 t & \text{if } 0 \le t \le t_1 \\ P_v(t_1) e^{-(t-t_1)/T_p} & \text{if } t_1 \le t \le T \end{cases} \tag{21}$$

where

$$\begin{aligned} d_1 &= [2c_p k^2 T_p^2 + (2c_p + b_p) V_0 k T_p] P_0 / V_0^2 \\ d_2 &= 2 c_p k^2 T_p P_0 / V_0^2 \end{aligned} \tag{22}$$

According to pre- and post-steady state, the static exponent α_s can be obtained. Then $P_v(t)$ can be calculated, from which T_p, d_1 and d_2 can be estimated. Finally, b_p, c_p are determined from d_1, d_2 using eqn. 22. It is therefore concluded that the CDSM is identifiable with a ramp voltage variation.

From this example, it is clearly observed that the identifiability of some models depends on the type or shape of the input signal.

4.2 Example 4: Xu and Mansour's model

In 1994 [14], Xu and Mansour developed a load model as follows:

$$\begin{aligned} P &= x P_{t2}(V) \\ T_p \dot{x} + x P_{t2}(V) &= P_s(V) \\ x(0) &= P_0 \\ P_{t2}(V) &= (V/V_0)^{\alpha_1} \end{aligned} \tag{23}$$

In a step test, $P_{t2}(V)$ and $P_s(V)$ jump from one value to another value at $t = 0$, then keep constant. In the Appendix, it is proven that the solution of eqn. 23 is:

$$\begin{aligned} P(t) &= C_2 e^{-t/T_{p2}} + P_s^s \\ C_2 &= P_0 P_{t2}^2 - P_s^s \\ T_{p2} &= T_p / P_{t2}^s \end{aligned} \tag{24}$$

where $()^s$ is the the value corresponding to V_s. It is easy to calculate α_s from V_0, V_s and P_0, P_s^s. With $P(t)$, one can estimate C_2 and T_{p2}. Then using

$$C_2 = P_0 (V_s/V_0)^{\alpha_t} - P_s^s$$

one can get α_t. Finally, T_p can be determined from α_t and T_{p2}. As a result, Xu and Mansour's model is uniquely identifiable with a step test.

5 Numerical identifiability

5.1 Problem description

Many parameter estimation algorithms, especially for nonlinear models, are based on minimisation of an index. The major procedure is to search the best parameter vector Z^* in the search space S, which minimises an error tunction E, i.e.

$$E^* = \underset{Z=Z^*, Z \in S}{\text{minimise}} E(Z) \tag{25}$$

The error function E is usually taken as a non-negative and monotonously increasing function of output error

$$\begin{aligned} E &= \int_{t_0}^{T} J(\|Y_m(t) - Y_c(t)\|) dt \quad \text{(continuous)} \\ E &= \sum_{j=0}^{N} J(\|Y_m(j) - Y_c(j)\|) \quad \text{(discrete)} \end{aligned} \tag{26}$$

where

$[t_0, T]$ is the observation interval

$J(e)$ monotonously increasing function

j the jth time sample

N the number of all samples

$()_m$ the measured (or true) values

$()_c$ computed values.

The most widely used forms of $J(e)$ are square function, absolute function, square root function or their combinations.

Even the model is linear, and leaving aside some highly exceptional cases, we may meet the difficuties associated with minimisation problems when attempting numerical identification of model parameters. In particular it is not known, whenever a numerical program provides a solution, whether the solution is unique (in which case the solution found is fully justified), or whether a finite (but more than one) or infinite number of solutions exist. In the latter cases the solution found is only one of many. The next step then is either to determine the set of possible solutions and conclude that the actual solution is one of the mathematical solutions found, or whenever possible to propose experimental procedures leading to a unique solution.

5.2 Numerical algorithm

In [7, 18] the authors have proposed a genetic algorithm-based parameter estimation (GABPE) method. The GABPE method has been applied to both composite induction motor load models and nonlinear input/output load models with satisfactory results, which show that the GABPE approach is robust, simple and powerful. Its ability to find the global optimum is specially useful for solving the numerical identifiability problem.

Based on a genetic algorithm, the main procedure involved in the approach for numerical identifiability is given below:

(i) Search one solution Z_0 (model parameters) with GABPE approach. Z_0 is supposed to be the global optimum or very close to the optimum; the corresponding error function is E_0.

(ii) Search the contour with constant function value E_0. The obtained points are denoted as Z_1, Z_2, ..., Z_m.

(iii) If the distance between Z_i and Z_0 is small enough, i.e.

$$d_i = \|Z_i - Z_0\|/\|Z_0\| < \varepsilon_i, \text{for all } i = 1, 2, \ldots, m$$

where ε is a small positive constant. Then, the solution Z_0 is considered to be unique or it is identifiable. Otherwise, it is unidentifiable.

5.3 Case studies

The exact model parameters of the speed dynamic induction motor model given in Section 3.1 are as follows:

$$M = 5 \times 314.16 = 1571 \quad s_{cr} = 0.0833$$
$$T_m = 1 \quad X' = 0.1959$$

Using GABPE, the parameters are estimated with the dynamic response data when the voltage $V(t)$ is stepping from 1.0 to 0.85 at t = 0. The *a priori* data of genetic algorithm are following:

population size = 50

code length = 8

crossover prob. = 0.9

mutation prob. = 0.05.

Table 1: Numerical identifiability example

No.	M	S_{cr}	T_m	X'
0	1415.26	0.0808	0.9980	0.1962
1	1835.16	0.0622	0.9980	0.1962
2	972.61	0.1168	0.9980	0.1971
3	1390.05	0.0826	0.9980	0.1962
4	852.11	0.1321	0.9980	0.1962

The obtained Z_0 and the contour points Z_1, Z_2, ..., Z_m are given in Table 1, where E_0 = 389.8. The results clearly show that the model is unidentifiable, the same as concluded in Section 3.1.

6 Conclusions

The identifiability problem of electric load models has been studied for the first time. The well known first-order speed dynamic induction motor model and the higher-order linear model are shown to be structurally unidentifiable. A composite dynamic–static model is proven to be unidentifiable with step test, but identifiable with a ramp voltage variation. That is, its identifiability depends on input type.

Identifiability is an important issue of load modelling and should be given more attention. There is a need for further work in this area, for example.

(i) Identifiability of the well-known 3rd-order electromechanical induction motor load model

(ii) Identifiability of other IO models with different inputs

(iii) Identifiability of ANN models

(iv) Numerical identifiability problem

(v) Identifiability under noisy situations.

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9 Appendixes

9.1 Proof of eqn. 21

In view of eqn. 20, the slope constant k is readily obtained:

$$k = (V_0 - V_s)/t_1 \tag{27}$$

Hence, during the time period $t \in [0, t_1]$, the transient function and its derivate are:

$$g_v(t) = g_v(V_0) + [c_p k^2 t^2 - (2c_p + b_p)V_0 kt]P_0/V_0^2$$
$$\dot{g}_v = [2c_p k^2 t - (2c_p + b_p)V_0 k]P_o/V_0^2 \tag{28}$$

It is noted that eqn. 14 is a linear system if the voltage (and hence g_v) is known. As a result, we have

$$P_v(t) = \int_0^t e^{-(t-\tau)/T_p}\dot{g}_v(\tau)d\tau \tag{29}$$

Substitution of eqn. 28 into eqn. 29 yields

$$P_v(t) = e^{-t/T_p}\int_0^t e^{\tau/T_p}[2c_p k^2 \tau - (2c_p + b_p)V_0 k]d\tau \tag{30}$$

To make the integration easier, let $\rho = \tau/T_p$; then

$$\int_0^t e^{\tau/T_p}\tau d\tau = T_p^2 \int_0^{t/T_p} e^{\rho}\rho d\rho$$
$$= T_p^2[e^{\rho}(\rho - 1)]\Big|_0^{t/T_p}$$
$$= T_p^2[e^{t/T_p}(t/T_p - 1) + 1] \tag{31}$$

Substituting the above equation into eqn. 30, integrating the another term in eqn. 30, and making some combination, one can obtain

$$P_v(t) = P_0/V_0^2[2c_p k^2 T_p^2(t/T_p - 1 + e^{-t/T_p})$$
$$- (2c_p + b_p)V_0 k T_p(1 - e^{-t/T_p})]$$
$$= P_0/V_0^2[(2c_p k^2 T_p^2 + (2c_p + b_p)V_0 k T_p)(e^{-t/T_p} - 1)$$
$$+ 2c_p k^2 T_p t] \tag{32}$$

This will lead to eqn. 21.

9.2 Proof of eqn. 24

The proof begins by supposing

$$x(t) = C_3 e^{-\lambda t} + C_4 \tag{33}$$

where λ is a positive constant. As $t \to \infty$, x reaches a new steady state. Consequently, eqn. 23 becomes:

$$0 + x_\infty P_{t2}^s = P_s^s$$

that is:

$$x_\infty = P_s^s/P_{t2}^s \tag{34}$$

It is clear from eqns. 33 and 34 that:

$$C_4 = x_\infty = P_s^s/P_{t2}^s$$
$$C_3 + C_4 = x_0 = P_0 \tag{35}$$

Substitution of eqn. 33 into eqn. 23 results in

$$(P_{t2} - \lambda T_p)C_3 e^{-\lambda t} = P_s - C_4 P_{t2}$$

Eqn. 35 tells us that:

$$C_4 P_{t2}^s = P_s^s$$
$$P_s^s - C_4 P_{t2}^s = 0 \tag{36}$$

hence:

$$P_{t2} - \lambda T_p = 0$$
$$\lambda = P_{t2}/T_p = 1/T_{p2} \tag{37}$$

Substituting eqns. 33, 36 and 37 into $P(t)$, one obtains

$$P(t) = xP_{t2}^s$$
$$= C_3 P_{t2}^s e^{-t/T_{p2}} + C_4 P_{t2}^s$$
$$= C_2 e^{-t/T_{p2}} + P_s^s \tag{38}$$

where

$$C_2 = C_3 P_{t2}^s = (P_0 - C_4)P_{t2}^s$$
$$= P_0 P_{t2}^s - C_4 P_{t2}^s$$
$$= P_0 P_{t2}^s - P_s^s \tag{39}$$

Therefore, eqn. 24 has been proven.